5. COMPLEX NUMBERS

Consider the quadratic equation, $x^2 - 4 = 0$.

We have $x^2 = 4 \Rightarrow x = \pm\sqrt{4} = \pm 2$. Hence, $x = 2$ *and* $x = -2$ are the solutions of the quadratic equation.

Consider another quadratic equation, $x^2 + 4 = 0$

Here, we have $x^2 = -4 \Rightarrow x = \pm\sqrt{-4} = \pm\sqrt{-1\times 4} = \pm\sqrt{-1}\times\sqrt{4} = \pm i2$. The quantity i is introduced from the word iota.

Standard form of a complex number: $z = x + iy$, where x and y are real numbers and x is known as real part of z and y is known as imaginary part of z.

Integral powers of i

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = i^2.i = (-1)i = -i$

$i^4 = i^2.i^2 = (-1)(-1) = 1$

Note:

i. $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + 1 = i - 1 - i + 1 = 0$

ii. $i + i^2 + i^3 + i^4 + \ldots + i^{99} = \left(i + i^2 + i^3 + i^4\right) + \left(i^5 + i^6 + i^7 + i^8\right) + \ldots + \left(i^{93} + i^{94} + i^{95} + i^{96}\right) + i^{97} + i^{98} + i^{99}$

$= \left(i + i^2 + i^3 + i^4\right) + i^4\left(i + i^2 + i^3 + i^4\right) + \ldots + i^{92}\left(i + i^2 + i^3 + i^4\right) + i^{96}\left(i + i^2 + i^3\right)$

$= 0 + 1(0) + \ldots + \left(i^4\right)^{23}\times 0 + \left(i^4\right)^{24}(i + -1 + -i)$

$= 0 + 0 + \ldots + 0 + (1)^{24}(-1)$

$= 1(-1) = -1$

iii. $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

iv. $i^{-39} = \frac{1}{i^{39}} = \frac{1}{i^{38}i} = \frac{1}{\left(i^2\right)^{19} i} = \frac{1}{(-1)i} = \frac{1}{-i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = -i$

Note:

1. $(-a-b)^2 = \left[-(a+b)\right]^2 = (a+b)^2$
2. $(-a-b)^3 = \left[-(a+b)\right]^3 = -(a+b)^3$

STANDARD FORM

$z = x+iy,\ x, y \in R$ is the standard form of a complex number. Here x is known as real part of z [$\mathrm{Re}(z)$] and y is known as imaginary part of z [$\mathrm{Im}(z)$].

SET-BUILDER FORM

Let C be the set of complex numbers and of $C = \left\{(x, y) : x, y \in R \text{ and } i = \sqrt{-1}\right\}$.

ALGEBRA OF COMPLEX NUMBERS

1. **Addition**: If $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers, then

   $z_1 + z_2 = a+ib+c+id = (a+c)+i(b+d)$

   E.g.: $z_1 = 2+i3\,; z_2 = 5+i2$

   $z_1 + z_2 = 2+i3+5+i2 = (2+5)+i(3+2) = 7+i5$

2. **Subtraction**: If $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers, then

   $z_1 - z_2 = a+ib-(c+id) = (a-c)+i(b-d)$

   E.g.: $z_1 = 2+i3\,; z_2 = 5+i2$

   $z_1 - z_2 = 2+i3-(5+i2) = (2-5)+i(3-2) = -3+i$

3. **Multiplication:** If $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers, then

   $z_1 z_2 = (a+ib)(c+id) = ac+iad+ibc+i^2bd = (ac-bd)+i(ad+bc)$

   E.g.: $z_1 = 2+i3\,; z_2 = 5+i2$

   $z_1 z_2 = (2+i3)(5+i2) = (2)(5)+(2)(i2)+(i3)(5)+(i3)(i2) = 10+4i+15i+6i^2$

   $= (10-6)+19i = 4+i19$

4. **Division**: If $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers, then

   $$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

   E.g.: $z_1 = 2+i3\,; z_2 = 5+i2$

   $$\frac{z_1}{z_2} = \frac{2+i3}{5+i2} = \frac{(2+i3)}{(5+i2)} \times \frac{(5-i2)}{(5-i2)} = \frac{10-4i+15i-6i^2}{25-4i^2} = \frac{10+6+11i}{25+4} = \frac{16}{29} + i\frac{11}{29}$$

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Properties of addition:

1. **Closure property** : If z_1 and z_2 be any two complex numbers, then z_1+z_2 is also a complex number.
2. **Commutative property** : If z_1 and z_2 be any two complex numbers, then $z_1+z_2=z_2+z_1$.
3. **Associative property** : If z_1 z_2 and z_3 be any two complex numbers, then $z_1+(z_2+z_3)=(z_1+z_2)+z_3$.
4. **Existence of additive identity**: For every complex number z, there exists a zero complex number, such that $z+0=z=0+z$
5. **Existence of additive inverse**: For every complex number z_1, there exists a zero complex number z_2, such that $z_1+z_2=0$, a zero complex number, then z_1 is the additive inverse of z_2 and z_2 is the additive Inverse of z_1.

Properties of Multiplication

1. **Closure property** : If z_1 and z_2 be any two complex numbers, then z_1z_2 is also a complex number.
2. **Commutative property** : If z_1 and z_2 be any two complex numbers, then $z_1z_2=z_2z_1$.
3. **Associative property** : If z_1 z_2 and z_3 be any two complex numbers, then $z_1(z_2z_3)=(z_1z_2)z_3$.
4. **Existence of multiplicative identity**: For every complex number z, there exists a zero complex number, such that $z.1=z=1.z$, where $1=1+i0$.
5. **Existence of multiplicative inverse**: For every non-zero complex number z, there exists a complex number $\frac{1}{z}$ or z^{-1} is known as multiplicative inverse of z such that $z.\frac{1}{z}=1$.
6. **Distributive property**: z_1 z_2 and z_3 be any two complex numbers, then
   a) $z_1(z_2+z_3)=z_1z_2+z_1z_3$ (Left distributive law)
   b) $(z_1+z_2)z_3=z_1z_3+z_2z_3$ (Right distributive law)

Modulus of a complex number.

Let $z=a+ib$ be a complex number, then modulus of z is the non-zero real number $\sqrt{a^2+b^2}$ is known as modulus of z. It is denoted by mod z or $|z|$.

E.g.: Let $z=3+i4$, then $|z|=\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5$

Properties

If z_1 and z_2 be any two complex numbers, then

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, |z_2| \neq 0$

Conjugate of a complex number

Let $z = a + ib$ be a complex number, then conjugate of z is denoted by $\bar{z} = a - ib$. i.e., if two complex numbers are differ by the sign of their imaginary parts, they are known as conjugate complex numbers.

z	$\bar{z}$
$2+i3$	$2-i3$
$2-i3$	$2+i3$
$-2+i3$	$-2-i3$
$-2-i3$	$-2+i3$

Properties

If z_1 and z_2 be any two complex numbers, then

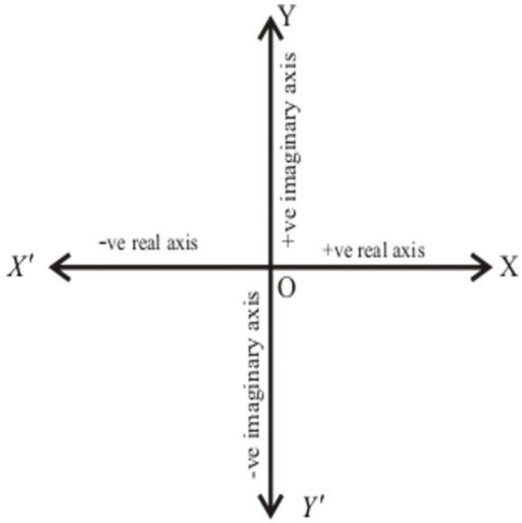
1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
3. $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$
4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

Note: For every complex number z, $z.\bar{z} = |z|^2$.

Argand plane and polar representation

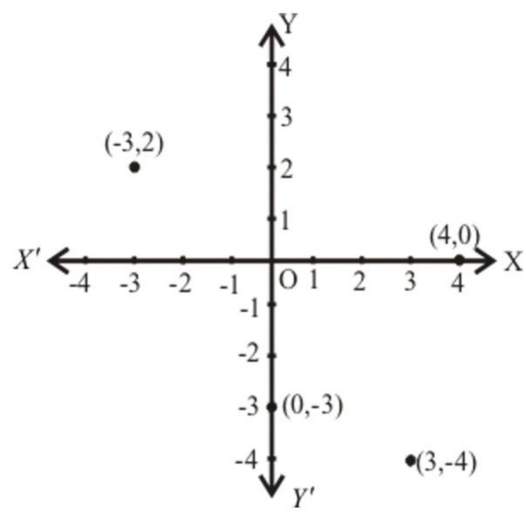
The representation of complex numbers as points in the plane is known as the Argand diagram. The plane having a complex number assigned to each of its points is called complex plane or Argand plane or Gaussian plane.

It consists of two mutually perpendicular lines. The horizontal line is known as real axis and the vertical line is known as imaginary axis.



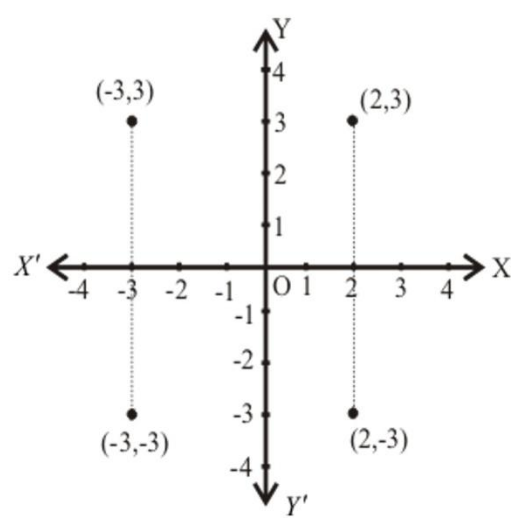
Represent the following complex numbers in a complex plane:

1. $3-i4$
2. $-3+i2$
3. $4(=4+i0)$
4. $-3i(=0-3i)$



Represent the following complex numbers and their conjugates in a complex plane:

5. $2+i3$
6. $-3+i3$



Polar form or Modulus-Amplitude form or Trigonometric form of a complex number.

Let P(x,y) be a point in the complex plane representing the non-zero complex number $z = x + iy$. Let θ radian be the angle made by the directed line segment OP with the positive real axis OX in the anticlockwise direction. Let OP = r, is known as modulus or absolute value of the complex number z denoted by $|z|$ or $\mathrm{mod}(z)$.The pair (r,θ) is known as polar coordinates of the point P. θ is called amplitude or argument of the complex number, denoted by $\arg z$ or $amp\, z$.

Let $z \equiv x + iy = r(\cos\theta + i\sin\theta)$ .... (1)

Equating the real and imaginary parts,

$r\cos\theta = x$ ....(2)

$r\sin\theta = y$ ....(3)

$(2)^2 + (3)^2$ $we\ have$

$r^2\left(\cos^2\theta + i\sin^2\theta\right) = x^2 + y^2$

$r^2 \times 1 = x^2 + y^2$

$\boxed{\therefore\ r = \sqrt{x^2 + y^2}}$, is the modulus of z.

$(3) \div (2) \Rightarrow$

$$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$$

$\boxed{\tan\theta = \frac{y}{x}} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$, is known as argument or amplitude of z.

Short cut method to find $\arg z$

x	y	$\arg z$ **lies in**	$\arg z =$
$+ve$	$+ve$	Quadrant I	θ
$-ve$	$+ve$	Quadrant II	$\pi - \theta$
$-ve$	$-ve$	Quadrant III	$\theta - \pi$
$+ve$	$-ve$	Quadrant IV	$-\theta$

Using the above facts, we can easily find the polar forms of the complex numbers in the four quadrants.

Find the polar form of the following complex numbers:

1. $1 + i\sqrt{3}$
2. $-1 + i\sqrt{3}$
3. $-1 - i\sqrt{3}$
4. $1 - i\sqrt{3}$

1. $1+i\sqrt{3}$

Let $1+i\sqrt{3}=r\left(\cos\theta+i\sin\theta\right)$ .... (1)

Here $x=1$ *and* $y=\sqrt{3}$

$$r=\sqrt{x^2+y^2}=\sqrt{1^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$$

$$\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{1}=\sqrt{3}=\tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow\theta=\frac{\pi}{3}$$

Since x is positive and y is positive, $\arg z$ lies in quadrant I

$$\therefore\ \arg z=\theta=\frac{\pi}{3}$$

In (1), we have $1+i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

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2. $-1+i\sqrt{3}$

Let $-1+i\sqrt{3}=r\left(\cos\theta+i\sin\theta\right)$ .... (1)

Here $x=-1$ *and* $y=\sqrt{3}$

$$r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$$

$$\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{-1}=-\sqrt{3}$$

$$If\ \tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$$

Since x is negative and y is positive, $\arg z$ lies in quadrant II

$$\therefore\ \arg z=\pi-\theta=\pi-\frac{\pi}{3}=\frac{3\pi-\pi}{3}=\frac{2\pi}{3}$$

In (1), we have $-1+i\sqrt{3}=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

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3. $-1-i\sqrt{3}$

Let $-1-i\sqrt{3}=r\left(\cos\theta+i\sin\theta\right)$ .... (1)

Here $x=-1$ *and* $y=-\sqrt{3}$

$$r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+\left(-\sqrt{3}\right)^2}=\sqrt{4}=2$$

$$\tan\theta=\frac{y}{x}=\frac{-\sqrt{3}}{-1}=\sqrt{3}$$

$$If\ \tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$$

Since x is negative and y is negative, $\arg z$ lies in quadrant III

$$\therefore\ \arg z=\theta-\pi=\frac{\pi}{3}-\pi=\frac{\pi-3\pi}{3}=\frac{-2\pi}{3}$$

In (1), we have $-1-i\sqrt{3}=2\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$

But $\cos\left(-\theta\right)=\cos\theta$ *and* $\sin\left(-\theta\right)=-\sin\theta$

$$\therefore\ -1-i\sqrt{3}=2\left(\cos\frac{2\pi}{3}+i\left(-\sin\frac{2\pi}{3}\right)\right)=2\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$$

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4. $1-i\sqrt{3}$

Let $1-i\sqrt{3}=r\left(\cos\theta+i\sin\theta\right)$ .... (1)

Here $x=1$ *and* $y=-\sqrt{3}$

$$r=\sqrt{x^2+y^2}=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=\sqrt{4}=2$$

$$\tan\theta=\frac{y}{x}=\frac{-\sqrt{3}}{1}=-\sqrt{3}$$

$$If\ \tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$$

Since x is positive and y is negative, $\arg z$ lies in quadrant IV

$\therefore \arg z = -\theta = -\frac{\pi}{3}$

In (1), we have $1 - i\sqrt{3} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

But $\cos(-\theta) = \cos\theta$ *and* $\sin(-\theta) = -\sin\theta$

$\therefore -1 - i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\left(-\sin\frac{\pi}{3}\right)\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

Solution of a quadratic equation having imaginary roots.

In the quadratic equation, $ax^2 + bx + c = 0, a \neq 0,$ the quantity $b^2 - 4ac$ is known as discriminant, be it discriminates the nature of the roots of the quadratic equation. If $b^2 - 4ac < 0$, then the roots are imaginary. i.e., conjugate complex numbers.

E.g.:

1. Solve: $2x^2 + x + 3 = 0$

   Here $a = 2;\ b = 1;\ c = 3$

   $D = b^2 - 4ac = 1^2 - 4\times2\times3 = 1 - 24 = -23 < 0$

   $\therefore\ x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-23}}{2\times2} = \frac{-1 \pm i\sqrt{23}}{4}$

2. Solve: $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

   $\sqrt{2}x^2 + x + \sqrt{2} = 0$

   Here $a = \sqrt{2};\ b = 1;\ c = \sqrt{2}$

   $D = b^2 - 4ac = 1^2 - 4\times\sqrt{2}\times\sqrt{2} = 1 - 8 = -7 < 0$

   $\therefore\ x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\times\sqrt{2}} = \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}$

3. Solve: $x^2 + x + \frac{1}{\sqrt{2}} = 0$

   Here $a = 1;\ b = 1;\ c = \frac{1}{\sqrt{2}}$

   $D = b^2 - 4ac = 1^2 - 4\times1\times\frac{1}{\sqrt{2}} = 1 - \frac{2\times\sqrt{2}\sqrt{2}}{\sqrt{2}} = 1 - 2\sqrt{2} = -\left(2\sqrt{2} - 1\right) < 0$ [Note this]

$$\therefore\ x=\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{-\left(2\sqrt{2}-1\right)}}{2\times1}=\frac{-1\pm i\sqrt{2\sqrt{2}-1}}{2}$$

Square root of a complex number

Find the square root of the complex number:

1. $-15-8i$

   Let $\sqrt{-15-8i}=x+iy$ .... (1)

   Squaring we have,

   $-15-8i=(x+iy)^2=x^2+i2xy+i^2y^2=\left(x^2-y^2\right)+i2xy$

   Equating the real and imaginary parts, we have

   $x^2-y^2=-15$ ....(2)

   $2xy=-8$ ....(3)

   We know that $\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+4x^2y^2=(-15)^2+(-8)^2=225+64=289$

   $\therefore\ x^2+y^2=\sqrt{289}=17$ ....(4)

   (2) +(4) we have,

   $x^2-y^2=-15$

   $x^2+y^2=17$

   $2x^2=2$

   $x^2=1\Rightarrow x=\pm1$

   In (4), we have, $1^2+y^2=17\Rightarrow y^2=17-1=16\Rightarrow y=\sqrt{16}=\pm4$

   Since $2xy=-8$,

   When $x=1, y=-4$ and when $x=-1, y=4$

   $\therefore\ 1-4i$ and $-1+4i$ are the square roots.

Find the square root of the complex number:

2. $1+i$

   Let $\sqrt{1+i}=x+iy$ .... (1)

   Squaring we have,

   $1+i=(x+iy)^2=x^2+i2xy+i^2y^2=\left(x^2-y^2\right)+i2xy$

   Equating the real and imaginary parts, we have

   $x^2-y^2=1$ ....(2)

$2xy = 1$ ....(3)

We know that $\left(x^2+y^2\right)^2 = \left(x^2-y^2\right)^2 + 4x^2y^2 = 1^2 + 1^2 = 2$

$\therefore\ x^2 + y^2 = \sqrt{2}$ ....(4)

(2) +(4) we have,

$$x^2 - y^2 = 1$$
$$x^2 + y^2 = \sqrt{2}$$
$$-----------$$
$$2x^2 = \sqrt{2} + 1$$
$$x^2 = \frac{\sqrt{2}+1}{2} \Rightarrow x = \pm\sqrt{\frac{\sqrt{2}+1}{2}}$$

In (4), we have, $\left(\sqrt{\frac{\sqrt{2}+1}{2}}\right)^2 + y^2 = \sqrt{2} \Rightarrow y^2 = \sqrt{2} - \frac{\sqrt{2}+1}{2} = \frac{2\sqrt{2}-\sqrt{2}-1}{2} = \frac{\sqrt{2}-1}{2}$

$\therefore\ y = \pm\sqrt{\frac{\sqrt{2}-1}{2}}$

Since $2xy = 1$,

When $x = \sqrt{\frac{\sqrt{2}+1}{2}}, y = \sqrt{\frac{\sqrt{2}-1}{2}}$ and when $x = -\sqrt{\frac{\sqrt{2}+1}{2}}, y = -\sqrt{\frac{\sqrt{2}-1}{2}}$

$\therefore\ \sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}$ and $-\sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}}$ are the square roots.

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